Embedding entropic algebras into modules

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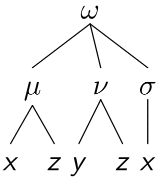
Theorem (D. Stanovský and M. Stronkowski)

An algebra without constants is quasi-linear iff it is quasiaffine

Branch decomposition

Example of the branch decomposition of a term



$t(x, z)$

$$\left\{ \begin{array}{c} (\omega,1) \\ | \\ (\mu,1) \\ | \\ x \end{array}, \begin{array}{c} (\omega,1) \\ | \\ (\mu,2) \\ | \\ z \end{array}, \begin{array}{c} (\omega,2) \\ | \\ (\nu,1) \\ | \\ y \end{array}, \begin{array}{c} (\omega,2) \\ | \\ (\nu,2) \\ | \\ z \end{array}, \begin{array}{c} (\omega,3) \\ | \\ (\sigma,1) \\ | \\ x \end{array} \right\}$$

$\mathrm{BD}(t)$

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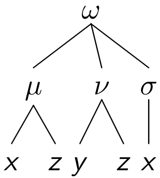

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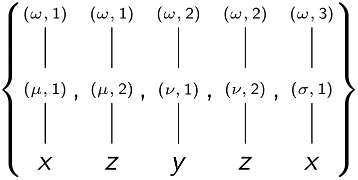

$\mathrm{BD}(t)$

Theorem (D. Stanovský and M. Stronkowski)

An algebra is quasi-linear iff it satisfies all quasi-identities

$$[t_1 \approx s_1 \wedge \cdots \wedge t_n \approx s_n] \to t_0 \approx s_0$$

for which the equality $\biguplus_{i=0}^{n} \mathrm{BD}(t_i) = \biguplus_{i=0}^{n} \mathrm{BD}(s_i)$ of multisets holds

What about commutative rings?

Problem

Characterize algebras embeddable into modules over commutative rings

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An algebra is entropic if it satisfies the identities

$$\mu(\nu(x_1^1, \dots, x_n^1), \dots, \nu(x_1^m, \dots, x_n^m)) \approx \nu(\mu(x_1^1, \dots, x_1^m), \dots, \mu(x_n^1, \dots, x_n^m))$$

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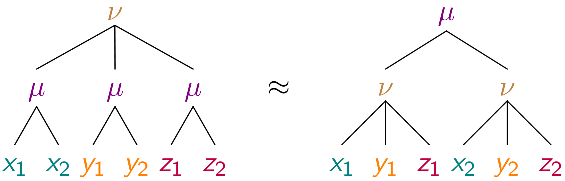

commutative rings?

Fact

Algebras embeddable into modules over commutative rings are entropic

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Algebras embeddable into modules over commutative rings are entropic

Specified Problem

Is it true that an algebra embeds into a module over a commutative ring iff it is quasi-linear and entropic?

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SubProblem

Is it true that an entropic algebra with one at least binary cancellative operation embeds into a module over a commutative ring?

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b - a branch, $[b]$ = all branches which differ from b only in order

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$$b = \begin{array}{c} (\omega, 1) \\ | \\ (\mu, 2) \\ | \\ x \end{array} \qquad [b] = \left\{ \begin{array}{c} (\omega, 1) \\ | \\ (\mu, 2) \\ | \\ x \end{array} , \begin{array}{c} (\mu, 2) \\ | \\ (\omega, 1) \\ | \\ x \end{array} \right\}$$

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Proposition

An algebra embeds into a module over a commutative ring iff it satisfies all quasi-identities

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for which the equality $\biguplus_{i=0}^{n} \mathrm{CBD}(t_i) = \biguplus_{i=0}^{n} \mathrm{CBD}(s_i)$ holds

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This is all Thank you!